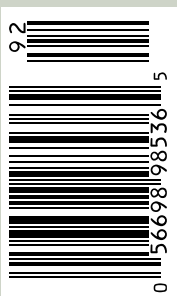
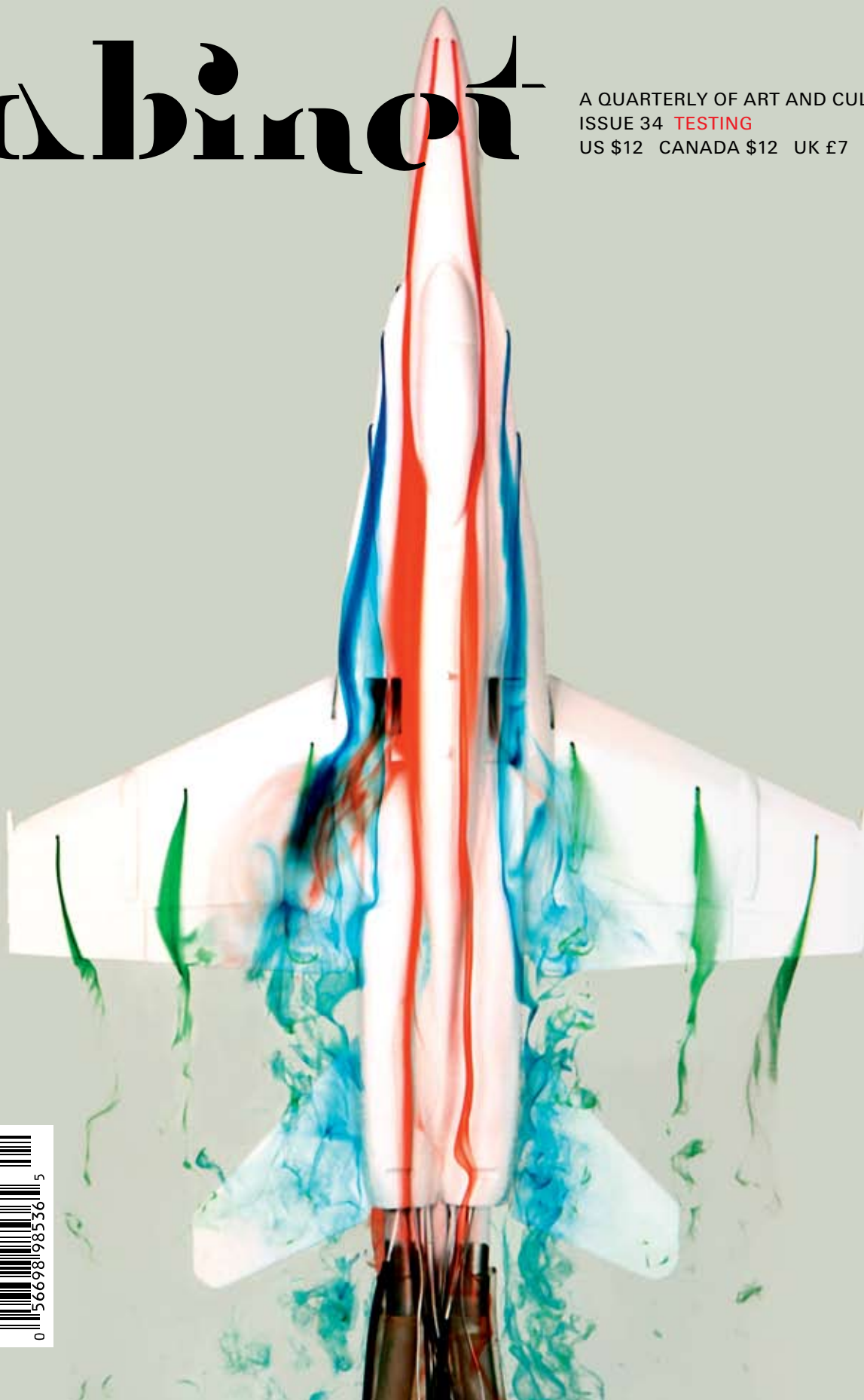


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## HUNTING A MATHEMATICAL SNARK

MARGARET WERTHEIM

*We have found the monster but it remains an enigma.*

—Mark Ronan

In 1940, the French mathematician Eli Cartan was speaking to a group of colleagues about his life's work on the foundations of the terrifyingly beautiful field known as group theory. "More than any other science," Cartan said, "mathematics develops through a sequence of consecutive abstractions. A desire to avoid mistakes forces mathematicians to find and isolate the essence of problems and the entities considered." The mathematician is compelled to wield a knife, paring back extraneous features until the mere bare bones of any concept are left. "Carried to an extreme," Cartan noted, "this procedure justifies the well-known joke that a mathematician is a scientist who knows neither what he is talking about nor whether what he is talking about exists or not."

Throughout its history, the movement of mathematics has been from the real to the surreal to the apparently absurd. If mathematics begins with things—things that can be counted, for example, like sheep and goats, or things that can be measured, like acres of land and ounces of gold—its historical unfolding has repeatedly followed a path of dematerialization, inexorably moving its focus from physical objects to an ethereal order that seems to stand, somehow, beyond *this* world. By stripping away the qualities of actual things, mathematicians become free to see the qualities of the ideas themselves—the idea of a "number," say, or of a "line"—and what is paradoxical here is that the components of this mathematical universe turn out to possess their own bewildering complexities. "Numbers" and "lines" and other apparently trivial mathematical entities have internal complications that on close inspection are almost unimaginably intricate. It is as if these ideas are imbued with their own innate anatomies; moreover, they are often related to one another in dazzlingly arcane ways. This unexpected complexity and interconnectedness leads many mathematicians to feel that they are engaged in the exploration of another world, a vast and magnificent continent populated by seemingly necessary "creatures," some of them no less mystical than unicorns or dragons and no less unlikely. Many branches of mathematics demonstrate this pattern, but the field that Cartan helped to develop—group theory—remains one of the most compellingly bizarre.

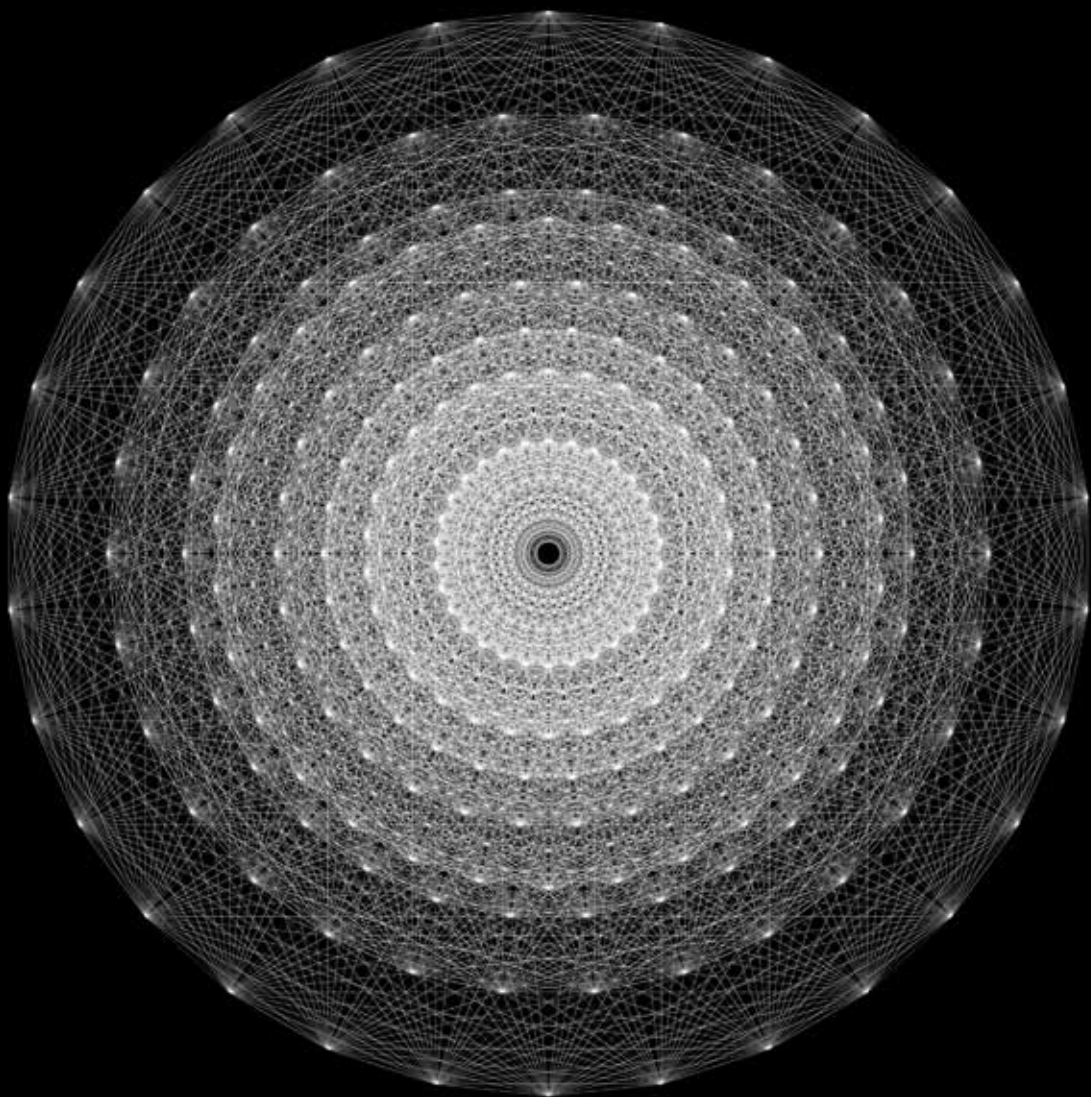
Through group theory, mathematicians in the 1970s came to understand that there might exist a structure

with 196,883 dimensions. As a mathematical object, it stood in a class of its own, a chimera that refused to be categorized according to any of the classifications group theorists had devised. It was the mathematical equivalent of Lewis Carroll's Snark, and the question that began to obsess them was whether or not it was real. They did not know if it existed, but they had begun to understand what some of its qualities might be. The name they had given it was "the Monster."

The story of the Monster begins simply enough. Imagine in your mind a square. Now think about rotating this shape so that at the end of the rotation the square appears in the same position as it started. Note that the key word here is *appears*, which means there are precisely four rotations you can make that will achieve this— $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ , and  $360^\circ$ . Any other rotation will leave the square skewed at the end compared with its starting position. These four specific rotations capture something fundamental about the structure of a square—they tell us that it has four-fold rotational symmetry. Now think about a pentagon—in this case the shape has five-fold rotational symmetry. Or a hexagon, whose rotational possibilities are six-fold. We can capture the essence of these symmetries by thinking about the sides of each polygon as a group of elements that can be transformed into one another by the "operation" of rotation. Mathematicians call this the rotational "group" of the form.

Another way of preserving the symmetry of a square is to reflect it in a mirror: a square, for example, looks the same on both sides of a mirror (unlike your left and right hands, which don't preserve mirror symmetry and are what is known as chiral.) Some forms have mirror symmetry, some have rotational symmetry, some have both. The totality is captured in what is known as the "symmetry group" of the form.

It turns out that we can do a similar exercise for three-dimensional forms. Think now of a cube. It has six faces, each of which has four sides. So there are twenty-four rotations that can be done on a cube that will bring it back to the same apparent position it started from. Mathematically, this means that underlying the structure of a cube is a rotational group with twenty-four elements. A cube can also be mirror-reflected. Take another highly regular three-dimensional form, the dodecahedron, which has twelve faces each of which is a pentagon: now there are sixty ways (twelve multiplied by five) this object can be rotated to bring it back into line with its starting position. Underlying the dodecahedron is thus a rotational group with sixty elements. It turns out that we can do this exercise with any finite

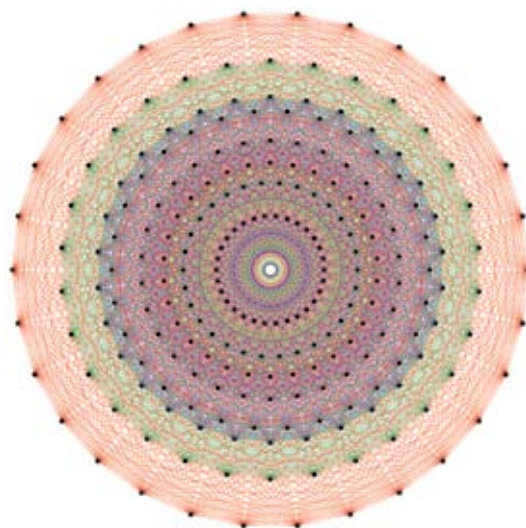


faceted shape in any number of dimensions and find a symmetry group that will capture the ways in which the object can be rotated or reflected so that it appears in the same position.

Such groups—which are collectively known as finite simple groups—turn out to have their own inherent classification system; they are arranged into families and sub-families that fall naturally together. The families are subject to a rigorous and formal taxonomy. Group theorists are in effect the botanists of this domain, studying the formal properties of each one, listing their different qualities and properties, and classifying the relationships between them. The groups themselves are infinite in number—the alphabet of symmetry is endlessly various (a fact that was important to prove)—but almost all of them fall into one of several major familial lines. Just as most living things fall into one or another of the categories of Eukarya, Bacteria, and Archaea, so finite simple groups can be marshaled into great lineages—all except for twenty-six aberrant outliers. Each of these “exceptional” groups fails to conform to any of the cladistic norms of the major group families and stands alone as a deviant. The Monster is the largest and strangest of this unruly flock.

It was in the early 1970s that mathematicians first began to understand that the Monster might exist. By 1960, they had classified many of the major group types in extensive tables, but at this point they knew about only five exceptions, all of which had been discovered in the nineteenth century. Then in 1963, two mathematicians, Walter Feit and John Thompson, developed a theorem that opened up the possibility of searching for more miscreants, and in 1966, Zvonimir Janko, working in Australia, found three more. To carry on our zoological analogy, this was like discovering three new types of mammals—marsupials, say, plus two other kinds that no one had ever imagined. The hunt was now on for more, and group theorists the world over set out to snare new exotics. By 1972, the number of exceptions had risen from five to twenty.

The first hint of the Monster actually came in triplicate. Bernd Fischer, who had been exploring the far reaches of group-land, discovered a new way in which groups could exhibit mirror symmetry on a vast scale. This bizarre and beautiful insight led him to a surprising new species that he tentatively titled  $M^{22}$ . Gigantic in size, it appeared to be a cross-section of something even larger, which was related to a group he had previously found called  $F_1^{24}$  (Fi stands for Fischer; being a mathematician is one of the surest ways to be immortalized). Fischer called this unseen behemoth  $M^{24}$  and



he speculated that there may be yet another group between these two giants—naturally, he called it  $M^{23}$ . All three groups were pure speculation; Fischer had not seen any of them directly, but his instincts told him they *ought* to exist. The English mathematical genius John Conway dubbed them the Baby Monster, the Middle Monster, and the Super-Monster. Soon it became clear that the Middle Monster was a chimera, and Conway settled on the Baby Monster and the Monster as his preferred terminology for the remaining two, a nomenclature that mathematicians have used ever since.

The story of the Monster and the history of symmetry groups in general is recounted with brio by mathematician Mark Ronan in his 2007 book *Symmetry and the Monster*. This is one of several books on the subject written in the last few years. Marcus Du Sautoy gives us a longer and more personal account in his 2008 *Symmetry*. It is funny how books on mathematics seem to come in flocks: few people outside the field had heard of Fermat’s Last Theorem before 1997, when there were suddenly three books on the subject published within a year. Ditto for the Riemann Hypothesis in 2003 (Du

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opposite and above: Computer representations of the exceptional finite group known as  $E_8$ , a smaller cousin of the Monster, marked by exquisite internal symmetry. The mathematics of symmetry groups also turns out to be central to the study of particle physics; in its “continuous,” rather than “finite” mode, group theory lies at the heart of quantum mechanics, and it is the properties of certain continuous “Lie groups” (named after the Norwegian mathematician Sophus Lie) that determine the arrangement of atoms in the periodic table. Physicist Antony Garrett Lisi’s recent and controversial “exceptionally simple theory of everything” proposes correspondence between each of the 248 symmetries of  $E_8$  and a different subatomic particle. Courtesy Jared Tarbell (opposite) and John Stembridge (above).

Sauty's *The Music of the Primes* was the standout here). Now it is the turn of symmetry. "Mathematical discoveries, like violets in the woods, have their seasons which no human can hasten or retard," said the legendary Carl Friedrich Gauss, and so it seems to be with mathematical books.

By the end of 1973, a small band of group theorists had begun to investigate the Monster's properties in earnest. They did not yet know if it existed but if it *did*, they wanted to recognize it when they saw it. The properties of each group are detailed in grid-like arrangements of numbers known as "character tables," so in the latter half of the 1970s some of the most brilliant mathematicians were literally cataloguing the character of a Monster.

Building on Fischer's work, they saw that one way of understanding the Baby Monster was as a permutation of 13,571,955,000 mirrors. In 1977, Fischer constructed a representation of this extraordinary situation on a computer, but the Monster—seen in this light—would have to be a permutation of 97,239,461,142,009,186,000 mirrors. No computer of the day could simulate such a complex array and, in 1978, Conway speculated he would not live to see its existence verified—it was too large, too complicated, and too strange. Ronan reports that Fischer calculated that doing just one of the necessary matrix multiplications would take half a year of computation time.

Then, out of the blue, on 14 January 1980, the American mathematician Robert Griess announced he had constructed the creature by hand, not using Fischer's method with the mirror reflections but another more mundane and painstaking approach. It remains a truly awesome feat, though Conway and others soon found easier ways with computers. What mattered was that the thing existed—at least insofar as *any* mathematical entity can be said to exist.

But here is the spooky thing: the Monster may encode the structure of our universe. There is tantalizing evidence that this extraordinary symmetry group may lie at the heart of physical reality as explained by string theory. In some versions of string theory, our universe is described as having twenty-six dimensions and there are mystically eerie resonances between this version of the theory and the formal characteristics of the Monster. There are whole columns of numbers in the Monster's character table that mirror properties of the mathematics that can also be used to describe the stringy structure of spacetime. No one knows if this is a coincidence or a genuine empirical sign, but in the 1980s the Princeton physicist Freeman Dyson expressed such a

vision: "I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the universe."

And this too is a pattern we find again and again in the history of mathematics: the study of objects leads to absurdly arcane abstractions, which then, miraculously, turn out to be manifest in the material world. It is a mystery that has been commented on by many of the leading physicists of the past century. "How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?" Einstein wrote. In seeking to understand how general relativity might be extended to include all the forces of nature, Einstein sought the advice of Eli Cartan and looked to the insights of group theory.

There is an argument to be made that symmetry is indeed nature's deepest physical principle. Given this, the discovery of another exception in the taxonomy of finite groups would be important news. Are there any more of these exotics lurking undetected? Most group theorists believe now that all the exceptions are known and that the Monster is by far the largest and strangest. In truth, we don't really know. It may well be that in some far corner of the vast continent of mathematics a new whole family of Snarks awaits discovery.